

Complexity phase diagram for interacting and long-range bosonic Hamiltonians

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Recent years have witnessed a growing interest in topics at the intersection of many-body physics and complexity theory. Many-body physics aims to understand and classify emergent behavior of systems with a large number of particles, while complexity theory aims to classify computational problems based on how the time required to solve the problem scales as the problem size becomes large. In this work, we use insights from complexity theory to classify phases in interacting many-body systems. Specifically, we demonstrate a “complexity phase diagram” for the Bose-Hubbard model with long-range hopping. This shows how the complexity of simulating time evolution varies according to various parameters appearing in the problem, such as the evolution time, the particle density, and the degree of locality. We find that classification of complexity phases is closely related to upper bounds on the spread of quantum correlations, and protocols to transfer quantum information in a controlled manner. Our work motivates future studies of complexity in many-body systems and its interplay with the associated physical phenomena.

Finding examples of quantum speedups over classical algorithms is a long-sought-after goal in quantum computing. In general, we do not know under what circumstances quantum computers offer exponential speedups over classical computers. Therefore, understanding the landscape of both quantum and classical computational complexity of various problems is an important goal in quantum computing. Understanding the quantum-classical divide is also of fundamental interest: whether a Turing machine can efficiently simulate natural phenomena is essentially a question about physics.

One way to understand the quantum-classical divide is to study classical simulability of quantum systems. This question has been addressed in the quantum circuit model through analysis of the simulability/non-simulability of different gate sets, yielding complexity classifications for families of circuits [1–13]. The same question may be asked for Hamiltonians [14, 15] and be studied under restrictions such as spatial locality [16, 17], which makes the role of evolution time more important.

In this work, we investigate when a quantum system evolving in time under a class of bosonic Hamiltonians is classically simulable and when it is not. Specifically, we study the problem of approximate sampling, which will be explained later on. We show how the complexity of this problem depends on the system parameters. For short times, we show that the system is efficiently classically simulable, whereas for longer times, the problem becomes classically intractable in the worst-case, indicative of a transition in the complexity of sampling. This

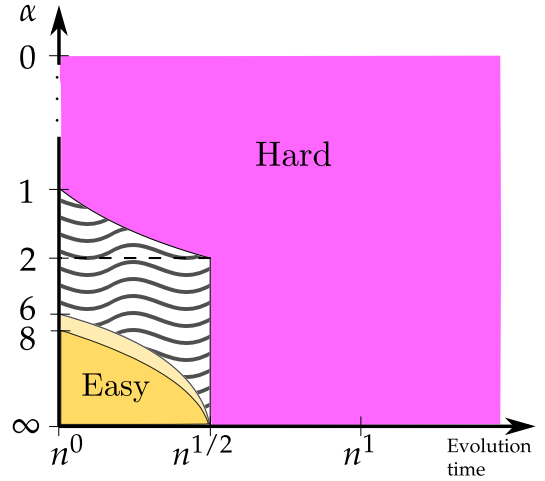


FIG. 1 (Color online). A slice of the complexity phase diagram for the long-range bosonic Hamiltonian in 2D when the initial inter-boson spacing is $L = \Theta(\sqrt{n})$. Colors represent whether the sampling problem is easy (yellow), hard (magenta), or not currently known (hatched). The X-axis parametrizes the exponent of n appearing in the evolution time and the Y-axis is α , the exponent characterizing the long-range nature of the hopping Hamiltonian (with scale $y = 1/\sqrt{\alpha}$ except for the point $\alpha = 0$). The darker, smaller easiness regime applies for all on-site interactions (Theorem 1.A), while the lighter, larger region applies for hard-core bosons (Theorem 1.B).

transition was investigated in Refs. [16, 17] under two different settings.

Reference [16] introduced these ideas by studying free bosons hopping between adjacent sites on a lattice. Here we generalize the work of Ref. [16] to systems with both on-site interactions and long-range hopping, using

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newer techniques. Specifically, we consider Hubbard interactions governed by strength V and hopping terms that decay with distance as a power law with exponent α . We assume some sparsity in the initial particle occupation, which is governed by a parameter β and the dimension D . We identify the timescale at which the complexity transition occurs, and understand its dependence on system parameters V, α, β , and D . This enables us to draw a phase diagram of complexity for the system, a slice of which is depicted in Fig. 1.

The Hamiltonian describes a Bose-Hubbard model in the nearest-neighbor limit, and is relevant to cold atom experiments that have been proposed as candidates for observing quantum computational supremacy [17–19]. In the hard-core limit, the bosons may be treated as spins, and the long-range hops translate to long-range interactions between these spins, which model quantum-computing platforms such as Rydberg atoms and trapped ions [20–24]. A phase diagram such as the one drawn in Fig. 1 clearly shows the “hard” and “easy” regions. One can only hope to find an exponential quantum speedup in the hard region, and, conversely, one cannot expect such a speedup in the easy region. Further, from a theoretical point of view, studying sampling complexity for long-range interacting systems provides a natural testbed for comparing long-range Lieb-Robinson bounds with protocols that aim to saturate the bounds [25, 26].

Setup.— Consider n bosons that initially occupy n distinct sites denoted by (i_1, \dots, i_n) . The initial state of the bosons can be written as $|\psi(0)\rangle = a_{i_1}^\dagger \dots a_{i_n}^\dagger |0\rangle$, where $|0\rangle$ is the vacuum state and a_i is a boson destruction operator on site i . The Hamiltonian is $H = \sum_{i,j} J_{ij}(t) a_i^\dagger a_j + \text{h.c.} + \sum_i f(n_i)$, with $n_i = a_i^\dagger a_i$. $J_{ij}(t)$ is a time-dependent hopping amplitude between sites i and j and can be long-ranged, falling off with the distance $d(i, j)$ as $|J_{ij}(t)| \leq 1/d(i, j)^\alpha$, while the on-site term J_{ii} may be unbounded. For the Bose-Hubbard Hamiltonian, we have $f(n_i) = V n_i(n_i - 1)/2$. In the $\alpha = 0$ limit, the system has all-to-all hops, while the $\alpha \rightarrow \infty$ limit corresponds to nearest-neighbor hops. This parameter governs how local or nonlocal the system is.

The aforementioned computational task of approximate sampling is to produce a sample that is approximately from the distribution \mathcal{D} induced by measuring the time-evolved state in the local boson-number basis. As in Ref. [16], we say the problem is *easy* if there exists a polynomial-time classical algorithm that samples from a distribution $\tilde{\mathcal{D}}$ that is $\epsilon = O(1/\text{poly}(n))$ -close in total variation distance to the target distribution \mathcal{D} [27] and *hard* if there cannot exist such an algorithm. If the initial state is easy to sample from, and there is some time at which the problem is worst-case hard, then there must be a timescale t_* such that the problem is always easy for time $t < \Theta(t_*)$, and hard for times $t > \Theta(t_*)$. In Ref. [16], the authors called this the transition timescale and derived lower and up-

per bounds on t_* . In this paper, we derive bounds on the transition timescale for more general (as compared to Ref. [16]) systems with long-range hops and on-site interactions. All hardness results in this paper rely on conjectures in complexity theory, such as the non-collapse of the polynomial hierarchy, anticoncentration, and average-to-worst case equivalence of certain approximation problems [10–13, 28].

Our easiness results come from bounds on how quickly information spreads [29–33]. Formally, for $\alpha > D + 1$, one can derive a Lieb-Robinson bound on the growth of the commutator $\|[A(t), B]\|$ of a Heisenberg time-evolved operator $A(t)$ with another local operator B supported on a region a distance r away. In the local limit $\alpha \rightarrow \infty$, information propagates within a causal “light-cone” region of radius $r = vt$, with exponentially decaying leakage outside the light-cone. However, in the nonlocal (finite α) regime, the tightest known bound on light-cone radius is $r \lesssim t^{(\alpha-D)/(\alpha-2D)}$ [33], with only a polynomial decay outside the light-cone. The polynomial decay of correlations is a barrier since the methods of Ref. [16] fail to work in this setting. We can give a sampling algorithm despite this barrier because of a powerful technique developed in the context of quantum simulation algorithms [34].

We consider bosonic systems where the lattice can be divided into K clusters C_1, \dots, C_K , containing b_1, \dots, b_K bosons, such that $b_1 + \dots + b_K = n$. We specifically consider sparsely distributed bosons in the initial state, for which each $b_i = O(1)$. Let the width L_i of a cluster C_i be defined as the minimum distance between a site outside the cluster and an occupied site inside the cluster $i_n \in C_i$, and let $L := \min_i L_i$. In Ref. [16], the authors considered a particular initial state where n bosons are distributed uniformly throughout the lattice of $m = \Theta(n^\beta)$ sites, with $b_i = \Theta(1)$ and $L_i = O((m/n)^{1/D}) = O(n^{(\beta-1)/D})$, with $\beta \geq 1$. For simplicity and for purposes of comparison, we use the same state to mention our main results, though we can generalize them to include other initial states as well.

Theorem 1.A (Easiness result for free and interacting bosons). *For all V (including $V = o(1)$ and $\omega(1)$ [35]) and $\alpha > D + 1$, we have $t_{\text{easy}} = \Omega(n^{c_{\text{easy}}})$. Here*

$$c_{\text{easy}} = \frac{\beta - 1}{D} \times \frac{\alpha - 2D\beta/(\beta - 1)}{\alpha - D + 1}. \quad (1)$$

If $c_{\text{easy}} < 0$, we have $t_{\text{easy}} = \Omega(\log n)$.

The above theorem is in fact true for any form of the on-site interaction $f(n_i)$. For hard-core bosons, we can show the following longer easiness timescale when $\alpha < \infty$, as illustrated in Fig. 1.

Theorem 1.B (Improved easiness result for hard-core bosons). *When $V \rightarrow \infty$ and $\alpha > D + 1$, then $t_{\text{easy}} = \Omega(n^{h_{\text{easy}}})$, with*

$$h_{\text{easy}} = \frac{\beta - 1}{D} \times \frac{\alpha - D(2\beta - 1)/(\beta - 1)}{\alpha - D}, \quad (2)$$

and $t_{\text{easy}} = \Omega(\log n)$ if $h_{\text{easy}} < 0$.

Theorem 2 (Hardness result). *For all V (including $V = o(1)$ and $\omega(1)$), we have $t_{\text{hard}} = O(n^{c_{\text{hard}}})$, where*

$$c_{\text{hard}} = \delta + \frac{\beta - 1}{D} \min[1, \alpha - D/2] \quad (3)$$

for any $\delta > 0$. The only exception to this result is the case of nearest-neighbor ($\alpha = \infty$) hard-core bosons in $D = 1$, for which $t_* = \infty$.

Easy-sampling timescale.— To derive t_{easy} , we approximate the time-evolved state by a product state on the clusters C_1, \dots, C_K and then show that the approximation is good for times $t < O(t_{\text{easy}})$. This product state approximation of the exact time-evolved state $|\psi(t)\rangle = U_t |\psi(0)\rangle$ is achieved by applying a recently developed spatial decomposition scheme [33, 34] to \tilde{U}_t , which we elaborate upon now.

Let H_R be the sum over all terms in the Hamiltonian supported completely in region R and let $XY = X \cup Y$ for any pair of regions. The time-evolved unitary is $U_{t_0, t_1}^R = \mathcal{T} \exp(-i \int_{t_0}^{t_1} H_R(s) ds)$. The decomposition scheme approximates a unitary acting on a region XYZ by a sequence of unitaries acting on XY, Y , and YZ : $U^{XYZ} \approx U^{XY}(U^Y)^\dagger U^{YZ}$, where Y separates regions X and Z . The accuracy of the approximation relies on the finite rate of information propagation in the system: the error is small for times t shorter than the time it takes for information to propagate from X to Z :

Lemma 3 ([33, 34]). *Let $X, Y, Z \subset \mathbb{R}^D$ be three disjoint regions such that X and XY are convex. Then for $\alpha > D + 1$,*

$$\|U_{0,t}^{XYZ} - U_{0,t}^{XY}(U_{0,t}^Y)^\dagger U_{0,t}^{YZ}\| \leq O((e^{vt} - 1)\Phi(X)\xi_\alpha(\ell)),$$

with $\xi_\alpha(\ell) = (16/(1 - \gamma))^\alpha \ell^{-\alpha+D+1} + e^{-\gamma\ell}$. Here the distance between the unitaries is measured in the operator norm, $v > 0$ is a Lieb-Robinson velocity, $\Phi(X)$ is the area of the boundary of X , γ can be chosen arbitrarily in the range $(0, 1)$, and $\ell = \text{dist}(X, Z)$ is the minimum distance between any pair of sites in X and Z .

In our application, the Lieb-Robinson velocity v that enters Lemma 3 is $O(n)$ in general instead of $O(1)$, since the hopping term $a_i^\dagger a_j$ has norm $\Theta(n)$ when restricted to the n -boson subspace. We deal with this by truncating the Hilbert space to have only $b + 1$ bosons on each site ($b = \max b_i$). The modified Hamiltonian H' after this truncation only has terms of norm $O(b) = O(1)$, giving an effective Lieb-Robinson velocity $v = O(1)$ for states close to the initial state [36]. We first prove an easiness result for this modified Hamiltonian H' with some polynomially small variation distance error (Lemma 4). Next, we bound the extra error caused by the evolution due to H' instead of H [37, 38]. This additional error leads to the full Hamiltonian H having a slightly smaller easiness timescale t_{easy} than that of H' .

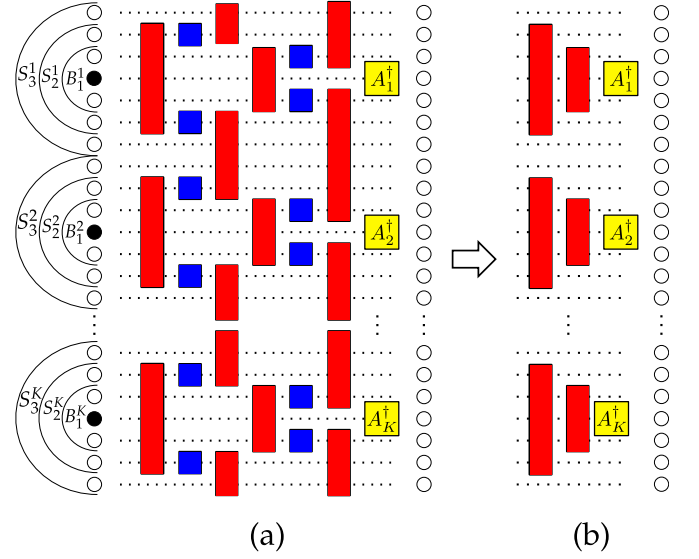


FIG. 2 (Color online). (a) Decomposition of the unitary $U_{t_1, t_2} U_{0, t_1}$ followed by (b) pushing the commuting terms past A_i^\dagger to the vacuum $|0\rangle$. Red boxes represent forward evolution and blue boxes backward evolution in time.

Lemma 4 (Decomposition error for H'). *For all V and $\alpha > D + 1$, when Lemma 3 is applied to decompose the evolution due to H' for time t , the error made (in 2-norm of the state) is*

$$\epsilon(t) \leq O\left(K(e^{vt_1} - 1)\xi_\alpha(\ell) \sum_{j=0}^{N-1} (r_0 + j\ell)^{D-1}\right) \quad (4)$$

$$\leq O(n(e^{vt/N} - 1)\xi_\alpha(\ell)NL^{D-1}), \quad (5)$$

where $N = t/t_1$ and $\ell \leq L/N$ can be chosen to minimize the error.

Lemma 4 can be derived from Lemma 3 without difficulty in the short-range case. To give an optimized error bound for the long-range case, we need to divide the time t into N slices. Here we give a unified proof in both the short- and long-range cases.

Proof of Lemma 4. Recall that within each cluster C_i , there is a group of bosons initially separated from the edge of the cluster by a region of width L_i . Let $A_i^\dagger(0) = \prod_{j \in C_i} a_{j_i}^\dagger$ be the creation operator for the group of bosons in a region. The initial state is $|\psi(0)\rangle = \prod_{i=1}^K A_i^\dagger(0) |0\rangle$. When evolved for short times, each creation operator $a_{j_i}^\dagger(t)$ is mostly supported over a small region around its initial position. Therefore, as long as these regions do not overlap, each operator approximately commutes, and the state is approximately separable.

Let A^i be the smallest ball upon which $A_i^\dagger(0)$ is supported. Let $B_0^i = A^i$ and denote its radius r_0^i , and define $r_0 = \max r_0^i$. B_k^i is a ball of radius $r_0^i + k\ell$ around A^i ,

where ℓ will be chosen to minimize the error. S_k^i is the shell $B_k^i \setminus B_{k-1}^i$ (see Fig. 2). The complement of a set X is denoted as X^c . We divide the evolution into N time steps between $t_0 = 0$ and $t_N = t$, and first show that the evolution is well-controlled for one time step from 0 to $t_1 = t/N$. We apply Lemma 3 K times, once for each cluster, letting $X = B_0^i$, $Y = S_1^i$ and Z be everything else:

$$U_{0,t_1} \approx U_{0,t_1}^{B_1^1} (U_{0,t_1}^{S_1^1})^\dagger U_{0,t_1}^{(B_0^1)^c} \quad (6)$$

$$\approx U_{0,t_1}^{B_1^1} (U_{0,t_1}^{S_1^1})^\dagger U_{0,t_1}^{B_2^1} (U_{0,t_1}^{S_2^1})^\dagger U_{0,t_1}^{(B_0^1 B_0^2)^c} \quad (7)$$

$$\approx U_{0,t_1}^{B_1^1} (U_{0,t_1}^{S_1^1})^\dagger \dots U_{0,t_1}^{B_K^1} (U_{0,t_1}^{S_K^1})^\dagger U_{0,t_1}^{(B_0^1 \dots B_0^K)^c}. \quad (8)$$

The total error is $O\left(\sum_{i=1}^K (e^{vt_1} - 1) \Phi(B_0^i) \xi_\alpha(\ell)\right) = O\left(K(e^{vt_1} - 1) r_0^{D-1} \xi_\alpha(\ell)\right)$. Applying the decomposed unitary to the initial state and pushing commuting terms through to the vacuum state, we get

$$U_{0,t_1} |\psi(0)\rangle \approx U_{0,t_1}^{B_1^1} A_1^\dagger \dots U_{0,t_1}^{B_K^1} A_K^\dagger |0\rangle = \left(\prod_{i=1}^K U_{0,t_1}^{B_i^1} A_i^\dagger\right) |0\rangle.$$

We can repeat the procedure for the unitary U_{t_1,t_2} , where $t_2 = 2t_1$. Now, the separating region Y will be S_2^i , so that $S_2^i \cap B_1^i = \emptyset$. Each such region still has width ℓ , but now the boundary of the interior is $\Phi(B_1^i) = O((r_0 + \ell)^{D-1})$. We get

$$U_{t_1,t_2} \approx \left(\prod_{i=1}^K U_{t_1,t_2}^{B_2^i} (U_{t_1,t_2}^{S_2^i})^\dagger\right) U_{t_1,t_2}^{(B_1^1 \dots B_1^K)^c}, \quad (9)$$

with error $O(K(e^{vt_1} - 1)(r_0 + \ell)^{D-1} \xi_\alpha(\ell))$. The unitaries supported on S_2^i and $(B_1^1 \dots B_1^K)^c$ commute with all the creation operators supported on sites B_1^i , giving $|\psi(t_2)\rangle \approx U_{t_1,t_2}^{B_2^1} U_{0,t_1}^{B_1^1} \dots U_{t_1,t_2}^{B_2^K} U_{0,t_1}^{B_1^K} |\psi(0)\rangle$. By applying this procedure a total of N times, once for each time step, we get the approximation $U_{0,t_N} |\psi(0)\rangle \approx U_{t_{N-1},t_N}^{B_N^1} \dots U_{0,t_1}^{B_1^1} \dots U_{t_{N-1},t_N}^{B_N^K} \dots U_{0,t_1}^{B_1^K} |\psi(0)\rangle$. The total error in the state (in 2-norm) is

$$\epsilon \leq O\left(K(e^{vt_1} - 1) \xi_\alpha(\ell) \sum_{j=0}^{N-1} (r_0 + j\ell)^{D-1}\right) \quad (10)$$

$$= O\left(n(e^{vt_1} - 1) \xi_\alpha(\ell) N L^{D-1}\right), \quad (11)$$

proving Lemma 4. The last inequality comes from the fact that $K \leq n$ and that $r_0 + (N-1)\ell \leq \min L_i = L$. The latter condition ensures that the decomposition of the full unitary is separable on the clusters. \square

Proof of Theorem 1.B. For hard-core bosons, there is no error made in truncating the Hilbert space, and so $H = H'$. Therefore Lemma 4 already gives the full error in simulating evolution due to the hard-core Hamiltonian.

We now choose parameters ℓ and $N = t/t_1$ to minimize the error in Eq. (11). We take $t_1 = O(1)$, giving $N = \Theta(t)$. We should choose ℓ as large as possible while still ensuring that $r_0 + (N-1)\ell \leq L_i$ for each cluster, hence we take $\ell = (L - r_0)/N = O(L/N)$. The algorithm is an approximate sampler as long as the approximation error $O(n t L^{D-1} \xi_\alpha(L/t))$ derived in Lemma 4 is small. For long-range systems ($\alpha \in [D+1, \infty)$), the error bound is $O(n t^{\alpha-D} L^{2D-\alpha})$. This is polynomially small in n as long as $t < O(L^{(\alpha-2D)/(\alpha-D)} n^{-1/(\alpha-D)})$.

When the easiness timescale above is $o(1)$, we choose the parameter $N = 1$ so that there is only one time step in total. The contribution from the perimeter only comes from that of the original support of A^i , which is $O(r_0^{D-1})$. Now we get the error to be $O(n r_0^{D-1} (e^{vt} - 1) L^{-\alpha+D+1})$, which is small as long as $t < O(\log n)$. Finally, for nearest-neighbor systems ($\alpha \rightarrow \infty$), we choose $N = 1$ and $\ell = L$. The behavior of $\xi_\alpha(L)$ is now $\sim e^{-L}$. This gives an error bound $O(n e^{vt-L})$, which is exponentially small in n as long as $t < O(L)$.

Since the time-evolved state is approximately separable on the clusters C_1, \dots, C_K , each cluster can be classically simulated with a runtime that depends on the number of particles contained in each cluster. With a simulation algorithm like exact diagonalization, the runtime is polynomial in the number of basis states, $(|C_i| + b_i - 1) = O(|C_i|^{b_i})$. Therefore, as long as b_i is $O(1)$, the state can be classically sampled from in time polynomial in n and m . Finally, sampling from a state close in 2-norm ensures a small total variation distance between the associated distributions [39]:

$$\|\mathcal{D} - \tilde{\mathcal{D}}\| \leq \|\psi\rangle - |\phi\rangle\|_2 = \epsilon, \quad (12)$$

which completes the proof of Theorem 1.B. \square

The proof of Theorem 1.A is analogous and can be found in Ref. [37].

Sampling Hardness timescale.— To derive t_{hard} , we give protocols to simulate quantum circuits by setting the time dependent parameters $J_{ij}(t)$. By implementing either an arbitrary linear optical unitary for boson sampling [10] or a quantum circuit that is hard to simulate [19] in a specific time, we show that a general sampling algorithm cannot work for times $t \geq t_{\text{hard}}$.

For the interacting case ($V = \Omega(1)$), we can use the Hubbard interaction to implement a controlled-phase gate CPHASE $[\phi]$ and form a universal gate set. We use the dual rail encoding, by which we can implement any single-qubit gate and a two-qubit gate between adjacent logical qubits in $O(1)$ time [40]. In order to implement a hard-to-simulate circuit, we implement the constant-depth circuit of Ref. [19], which consists of only nearest-neighbor gates in a 2D grid. However, since our initial bosons (and hence the logical qubits) are far apart, we need to move the logical qubits by a distance $O(L)$ to bring them near each other. This can be achieved using the state transfer protocol of Ref. [26], using all the other

modes in a block of radius L as ancillas. This takes time $O(L, \min[L^{\alpha-D/2}])$, giving the timescale in Theorem 2. The argument for 1D is given in Ref. [37]. We also consider hard-core bosons in Ref. [37], for which the entangling gate is constructed differently, and which also feature an easiness result for the 1D nearest-neighbor case.

In the non-interacting case ($V = o(1)$), Aaronson and Arkhipov [10] gave evidence that the state obtained after applying a Haar-random linear-optical unitary is hard to sample from. They further showed that it is possible to implement an arbitrary linear-optical unitary with depth $O(n \log m)$ in the circuit model when there is no spatial locality. We are working in the Hamiltonian model, which differs from the circuit model since we allow simultaneous noncommuting terms in the Hamiltonian. We show in Ref. [37], using ideas from state transfer, that most linear-optical states of n bosons on m sites can be reached in time $\min[O(nm^{1/D}), \tilde{O}(nm^{\alpha/D-1/2})]$, which is faster when $\alpha < D/2$. Further, by observing that we only need to implement boson sampling on a polynomially growing number of bosons n^δ in $O(n^\delta L^D)$ modes, we can optimize the hardness timescale exponent to reach $\delta + \frac{\beta-1}{D} \min[1, \alpha - D/2]$ and prove Theorem 2 for free bosons.

Outlook.— We have mapped out the complexity of the long-range Bose-Hubbard model on a phase diagram (Fig. 1). One remaining open question is to deal with the regions of the phase diagram where we do not have a definitive easiness/hardness result. This quest of identifying lower and upper bound on timescales is closely related to a similar quest in Lieb-Robinson bounds for long-range interacting systems [25, 26]. Lieb-Robinson bounds place limits on the time it takes to transfer a state from one site to another. However, for long-range interactions, these bounds are not known to be saturable, meaning that we do not know of protocols that achieve state transfer that is as fast as allowed by the bounds. The easiness timescale can be improved via better Lieb-Robinson bounds and the hardness regime can be extended to lower timescales via faster state-transfer protocols, and hence having saturable Lieb-Robinson bounds could bridge the gap in the phase diagram.

One may also carry out a similar classification of complexity for other physically interesting Hamiltonians and Liouvillians describing systems with active measurement with or without feedback, such as random circuits with projective measurements on part of the system [41–44]. These systems are fertile grounds for investigating the connection of sampling complexity with physical phenomena such as thermalization (and the various ways it can break down) and scrambling. The connection to scrambling particularly deserves further study: the growth of out-of-time-ordered correlators, which diagnose scrambling in chaotic systems, is characterized by a “butterfly velocity”, a state-dependent Lieb-Robinson velocity [45]. Our work implies a bounded butterfly velocity for systems with spatially separate bosons, even when the Lieb-Robinson velocity for general operators is unbounded. This suggests that sampling complexity could be related to other measures such as circuit complexity, which have been shown to be related to measures of scrambling [46]. Such a relation would then allow one to get more rigorous results for complexity in high-energy physics by drawing on the literature in the field of quantum computational supremacy.

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SUPPLEMENTAL MATERIAL

Abstract: In this Supplemental Material, we justify the usage of Lemma 3 with $v = O(1)$ to prove Lemma 4. We also bound the error caused by evolution under a different Hamiltonian $H' \neq H$ when the on-site interaction strength V is finite and not a hard-core interaction. We expand upon the hardness results and complete the proof of Theorem 2 for interacting bosons, followed by the proof of Theorem 2 for free bosons.

S1. VALIDITY OF HHKL DECOMPOSITION FOR TRUNCATED HAMILTONIAN

In this section, we define the truncated Hamiltonian, and argue that it is possible to apply the HHKL decomposition lemma to H' with a Lieb-Robinson velocity of order $O(1)$. As mentioned in the main text, H' is a Hamiltonian that lives in the truncated Hilbert space of at most $b + 1$ bosons on each site. Let Q be a projector onto this subspace. Then let $H' = QHQ$. Time-evolution under this modified Hamiltonian H' keeps a state within the subspace since $[e^{-iQHQt}, Q] = 0$.

The Lieb-Robinson velocity only depends on the norm of terms in the Hamiltonian which couple lattice sites. On-site terms do not contribute, which can be seen by moving to an interaction picture [32]. Therefore, since no state has more than $b + 1$ bosons on any site within the image of Q , the maximum norm of coupling terms in H' is $\|Qa_i^\dagger a_j Q\| \leq b + 1$. Therefore, the Lieb-Robinson velocity is at most $O(b)$ instead of $O(n)$, and we can apply Lemma 3 to the evolution generated by the truncated Hamiltonian H' .

S2. CLOSENESS OF EVOLUTION UNDER H AND H' .

We next show that the states evolving due to H and H' are close, owing to the way the truncation works. This will enable us to prove Theorem 1.A. Suppose that an initial state $|\psi(0)\rangle$ evolves under two different Hamiltonians $H(t)$ and $H'(t)$ for time t , giving the states $|\psi(t)\rangle = U_t |\psi(0)\rangle$ and $|\psi'(t)\rangle = U'_t |\psi(0)\rangle$, respectively. Define $|\delta(t)\rangle = |\psi(t)\rangle - |\psi'(t)\rangle$ and switch to the rotating frame, $|\delta^r(t)\rangle = U_t^\dagger |\delta(t)\rangle = |\psi(0)\rangle - U_t^\dagger U'_t |\psi(0)\rangle$. Now taking the derivative,

$$i\partial_t |\delta^r(t)\rangle = 0 + U_t^\dagger H(t) U'_t |\psi(0)\rangle - U_t^\dagger H'(t) U'_t |\psi(0)\rangle \quad (S1)$$

$$= U_t^\dagger (H(t) - H'(t)) |\psi'(t)\rangle. \quad (S2)$$

The first line comes about because $i\partial_t U'_t = H'(t) U'_t$ and $i\partial_t U_t^\dagger = -U_t^\dagger H(t)$, owing to the time-ordered form of U_t .

Now, we can bound the norm of the distance, $\delta(t) := \|\delta(t)\| = \|\delta^r(t)\|$.

$$\delta(t) \leq \delta(0) + \int_0^t d\tau \|(H(\tau) - H'(\tau)) |\psi'(\tau)\rangle\| \quad (S3)$$

$$= \int_0^t d\tau \|(H(\tau) - H'(\tau)) |\psi'(\tau)\rangle\|, \quad (S4)$$

since $\delta(0) = 0$.

The next step is to bound the norm of $(H - H') |\psi'(\tau)\rangle$ (we suppress the time label τ in the argument of H and H' here and below). We use the decomposition obtained from Lemma 3: $|\psi'(\tau)\rangle = |\phi(\tau)\rangle + |\epsilon(\tau)\rangle$, where the state $|\phi(\tau)\rangle$ is a product state over clusters, and $|\epsilon(\tau)\rangle$ is an unknown error induced by the HHKL decomposition. We first show that $(H - H') |\phi(\tau)\rangle = 0$. Since $|\phi(\tau)\rangle$ is a product state of clusters, each of which is time evolved separately, boson number is conserved within each cluster. Therefore, each cluster has at most b bosons, and $Q |\phi(\tau)\rangle = |\phi(\tau)\rangle$. Furthermore, only the hopping terms in H can change the boson number distribution among the different clusters, and these terms move single bosons. This implies $H |\phi(\tau)\rangle$ has at most $b + 1$ bosons per cluster, and remains within the image of Q . Combining these observations, we get $H' |\phi(\tau)\rangle = QHQ |\phi(\tau)\rangle = H |\phi(\tau)\rangle$. This enables us to say that $(H - H') |\phi(\tau)\rangle = (H - QHQ) |\phi(\tau)\rangle = 0$. Equation (S4) gives us

$$\delta(t) \leq \int_0^t d\tau \|(H - H')(|\phi(\tau)\rangle + |\epsilon(\tau)\rangle)\| \quad (S5)$$

$$= \int_0^t d\tau \|(H - H') |\epsilon(\tau)\rangle\|, \quad (S6)$$

$$\leq \max_{|\eta\rangle \in Q} \|(H - H') |\eta\rangle\| \int_0^t d\tau \|\epsilon(\tau)\|. \quad (S7)$$

In the last inequality, we have upper bounded $\|(H - H') |\epsilon(\tau)\rangle\|$ by $\max_{|\eta\rangle \in Q} \|(H - H') |\eta\rangle\| \times \epsilon(\tau)$, where $\epsilon(\tau) := \|\epsilon(\tau)\|$. The quantity $\max_{|\eta\rangle \in Q} \|(H - H') |\eta\rangle\|$ can be thought of as an operator norm of $H - H'$, restricted to the subspace Q . It is enough to consider a maximization over states $|\eta\rangle$ in the subspace of Q because we know that the error term $|\epsilon(\tau)\rangle$ also belongs to this subspace.

Now, let us find $\max_{|\eta\rangle} \|(H - QHQ) |\eta\rangle\|$ for $|\eta\rangle \in Q$. Notice that for each term H_i in the Hamiltonian, the operator $H - QHQ$ contains $H_i - QH_iQ$, where the rightmost Q can be neglected since $Q |\eta\rangle = |\eta\rangle$. The on-site terms $\sum_i J_{ii} a_i^\dagger a_i + V n_i(n_i - 1)/2$ do not change the boson number. Therefore, they cannot take $|\eta\rangle$ outside the subspace of Q , and do not contribute to $(H - QHQ) |\eta\rangle$. The only contribution comes from hopping terms that change boson number, which we bound by $\left\| \sum_{i,j} J_{ij} a_i^\dagger a_j \right\|$.

The hopping term can be rewritten in the normal mode basis, $H_{\text{hop}} = \sum_i \omega_i b_i^\dagger b_i$, where ω_i 's are the normal mode frequencies. The norm of this is therefore $n \times \max_i \omega_i$. The normal mode frequencies are $\{\omega_i\}$, which are actually the eigenvalues of the $m \times m$ matrix $J_{ij} =: J$. We should bound the maximum eigenvalue of J . To this end, we use the Gershgorin circle theorem, which states that the maximum eigenvalue of J is bounded by the quantity $\max_i \sum_j J_{ij}$. Since J_{ij} is bounded by $r(i, j)^{-\alpha}$, we can show that when $\alpha > D + 1$, for a D -dimensional lattice, the sum $\sum_j J_{ij}$ converges: $\sum_j r(i, j)^{-\alpha} = O(1)$ (see Lemma 5 of Ref. [33]). This gives $\|H_{\text{hop}}\| \leq O(n)$. Therefore, $\max_{|\eta\rangle \in Q} \|(H - H') |\eta\rangle\| \leq O(n)$. We are now in a position to prove Theorem 1.

Proof of Theorem 1.A. We have

$$\delta(t) \leq O(n) \int_0^t d\tau \epsilon(\tau). \quad (S8)$$

We integrate the result from Lemma 4, and find that the total error made in truncating the Hamiltonian to H' and in decomposing the evolution due to H' is

$$\delta + \epsilon = \begin{cases} O(n^2 t^{\alpha-D+1} L^{-\alpha+2D}), & \alpha > 2D + \frac{2D}{\beta-1} = \frac{2D\beta}{\beta-1} \\ O(n^2 e^{vt-L}), & \alpha \rightarrow \infty, \text{ and} \\ O(n^2 (e^{vt} - 1) L^{-\alpha+D+1}), & D+1 \leq \alpha \leq \frac{2D\beta}{\beta-1}. \end{cases} \quad (S9)$$

As discussed in the proof of Theorem 1.B, we make one of two choices $N = O(t)$ or $N = 1$ in order to optimize the total error and the timescale. In the first case, we have made the choice $N = O(t)$, while in the second and third cases $N = 1$. This proves that the easiness timescale is, in various cases,

$$t_{\text{easy}} = \begin{cases} O\left(L^{\frac{\alpha-2D}{\alpha-D+1}} n^{\frac{-2}{\alpha-D+1}}\right), & \alpha > \frac{2D\beta}{\beta-1} \\ O(L), & \alpha \rightarrow \infty, \text{ and} \\ O(\log L), & D+1 \leq \alpha \leq \frac{2D\beta}{\beta-1}. \end{cases} \quad (\text{S10})$$

Rewriting this in terms of the exponent of n appearing in the timescale $n^{c_{\text{easy}}}$, we observe that

$$c_{\text{easy}} = \frac{\beta-1}{D} \times \frac{\alpha-2D}{\alpha-D+1} - \frac{2}{\alpha-D+1} \quad (\text{S11})$$

$$= \left(\frac{\beta-1}{D}\right) \frac{\alpha-2D-2D/(\beta-1)}{\alpha-D+1} \quad (\text{S12})$$

$$c_{\text{easy}} = \left(\frac{\beta-1}{D}\right) \frac{\alpha-2D\beta/(\beta-1)}{\alpha-D+1}. \quad (\text{S13})$$

Here we have used the fact that $L = \Theta(n^{\frac{\beta-1}{D}})$. □

S3. HARDNESS TIMESCALE FOR INTERACTING BOSONS

In this section we provide more details about how to achieve the timescales in Theorem 2. In the interacting case, almost any interaction is universal for BQP [47] and hence these results are applicable to general on-site interactions $f(n_i)$.

We first describe how a bosonic system with fully controllable local fields $J_{ii}(t)$, hoppings $J_{ij}(t)$, and a fixed Hubbard interaction $\frac{V}{2} \sum_i \hat{n}_i(\hat{n}_i - 1)$ can implement a universal quantum gate set. To simulate quantum circuits, which act on two-state spins, we use a dual-rail encoding. Using $2n$ bosonic modes, and n bosons, n logical qubits are defined by partitioning the lattice into pairs of adjacent modes, and a boson is placed in each pair. Each logical qubit spans a subspace of the two-mode Hilbert space. Specifically, $|0\rangle_L = |10\rangle$, $|1\rangle_L = |01\rangle$. We can implement any single qubit (2-mode) unitary by turning on a hopping between the two sites (X-rotations) or by applying a local on-site field (Z-rotations). To complete a universal gate set, we need a two-qubit entangling gate. This can be done, say, by applying a hopping term between two sites that belong to different logical qubits [40]. All these gates are achievable in $O(1)$ time when $V = \Theta(1)$.

For hardness proofs that employ postselection gadgets, we must ensure that the gate set we work with comes equipped with a Solovay-Kitaev theorem. This is the case if the gate set is closed under inverse, or contains an irreducible representation of a non-Abelian group [48]. In our case, the gate set contains single-qubit Paulis and hence has a Solovay-Kitaev theorem, which is important for the postselection gadgets to work as intended.

We will specifically deal with the scheme proposed in Ref. [19]. It applies a constant-depth circuit on a grid of $\sqrt{n} \times \sqrt{n}$ qubits in order to implement a random IQP circuit [11, 13] on \sqrt{n} effective qubits. This comes about because the cluster state, which is a universal resource for measurement-based quantum computation, can be made with constant depth on a two-dimensional grid.

For short-range hops ($\alpha \rightarrow \infty$), we implement the scheme in four steps as shown in Fig. S1. In each step, we move the logical qubits to bring them near each other and make them interact in order to effect an entangling gate. For short-range hopping, the time taken to move a boson to a far-off site distance L away dominates the time taken for an entangling gate. The total time for an entangling gate is thus $O(L) + O(1) = O(L)$.

For long-range hopping, we use the same scheme as in Fig. S1, but we use the long-range hopping to speed up the movement of the logical qubits. This is precisely the question of state transfer using long-range interactions/hops [25, 26]. In the following we give an overview of the best known protocol for state transfer, but first we should clarify the assumptions in the model. The Hamiltonian is a sum of $O(m^2)$ terms, each of which has norm bounded by at most $1/d(i, j)^\alpha$. Since we assume we can apply any Hamiltonian subject to these constraints, in particular, we may choose to apply hopping terms across all possible edges. This model makes it possible to go faster than the circuit model if we compare the time in the Hamiltonian model with depth in the circuit model. This power comes about because of the possibility of allowing simultaneous noncommuting terms to be applied in the Hamiltonian model.

The state transfer protocol of Ref. [26] shows such a speedup for state transfer and is described in Sec. S4. It applies a map $|1\rangle_1 \rightarrow \sum_{j \neq 1,2} \frac{1}{\sqrt{N-2}} |1\rangle_j \rightarrow |1\rangle_2$ using two steps, each of which takes time $O(L^\alpha / \sqrt{N-2})$, where $N-2$ is the

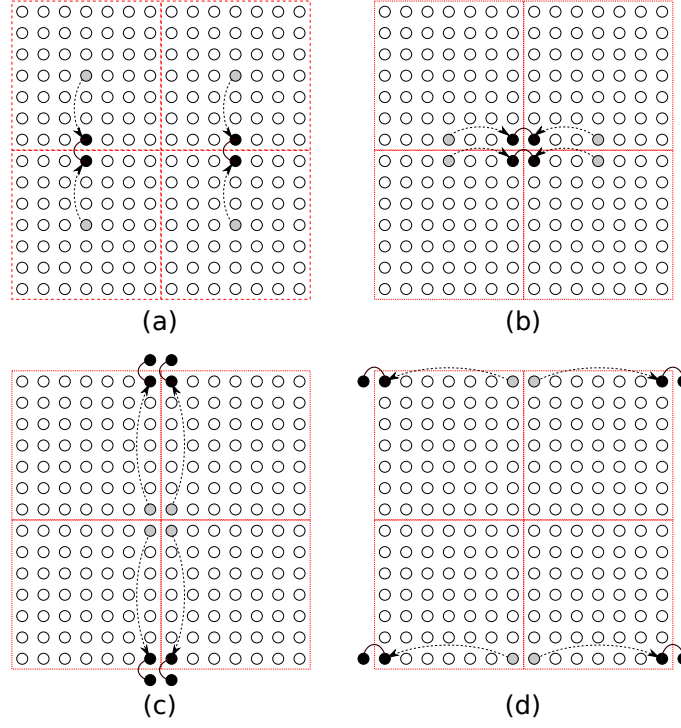


FIG. S1 (Color online). A protocol that implements the logical circuit of Ref. [19]. Each subfigure shows the location of the site that previously encoded the $|1\rangle$ state in gray. The current site that encodes the $|1\rangle$ state is in black. The site that encodes $|0\rangle$ is not shown but moves similarly as the $|1\rangle$ state. The distance traversed by each qubit is $L + L + 2L + 2L = 6L$.

number of ancillas used and L is the distance between the two furthest sites. Again, this speedup is because of the normalization in the Hamiltonian.

In our setting, we use the state transfer protocol to move the logical qubit faster in each step of the scheme depicted in Fig. S1. The number of ancillas used is $N - 2 = O(L^D)$, giving a state transfer time of $L^{\alpha-D/2}$. This time is faster than $O(L)$, the time it would take for the nearest-neighbor case, when $\alpha < \frac{D}{2} + 1$. Therefore, for 2D or higher, the total time it takes to implement a hard-to-simulate circuit is $\min[L, L^{\alpha-D/2}] + O(1)$, proving Theorem 2 for interacting bosons. When $\alpha < D/2$, the limiting step is dominated by the entangling gate, which takes time $O(1)$. Therefore for this case we only get hardness through boson sampling, which is discussed in Sec. S4. Note that when $t = o(1)$ and interaction strength is $V = \Theta(1)$, their effect is governed by $Vt = o(1)$, which justifies treating the problem for short times as a free-boson problem.

A. One dimension

In 1D with nearest-neighbor hopping, we cannot hope to get a hardness result for simulating constant depth circuits, which is related to the fact that one cannot have universal measurement-based quantum computing in one dimension. We change our strategy here. The overall goal in 1D is to still be able to simulate the scheme in Ref. [19] since it provides a faster hardness time (at the cost of an overhead in the qubits). The way this is done is to either (i) implement $O(n)$ SWAPs in 1D in order to implement an IQP circuit [11], or (ii) use the long-range hops to directly implement gates between logical qubits at a distance L away.

The first method takes time $O(\min[nL, L^{\alpha-1/2}n])$, where we again use the state transfer protocol to implement a SWAP by moving each boson within a cluster a distance $O(L)$.

The second method relies on the observation that when $\alpha \rightarrow 0$, the distinction between 1D and 2D becomes less clear, since at $\alpha = 0$, the connectivity is described by a complete graph and all hopping strengths are equal. Let us give some intuition for the $\alpha \rightarrow 0$ case. One would directly “sculpt” a 2D grid from the available graph, which is a complete graph on n vertices (one for every logical qubit) with weights w_{ij} given by $d(i, j)^{-\alpha}$. If we want to arrange qubits on a 1D path, we can assign an indexing to qubits in the 2D grid and place them in the 1D path in increasing order of their index. One may, in particular, choose a “snake-like indexing” depicted in Fig. S2.

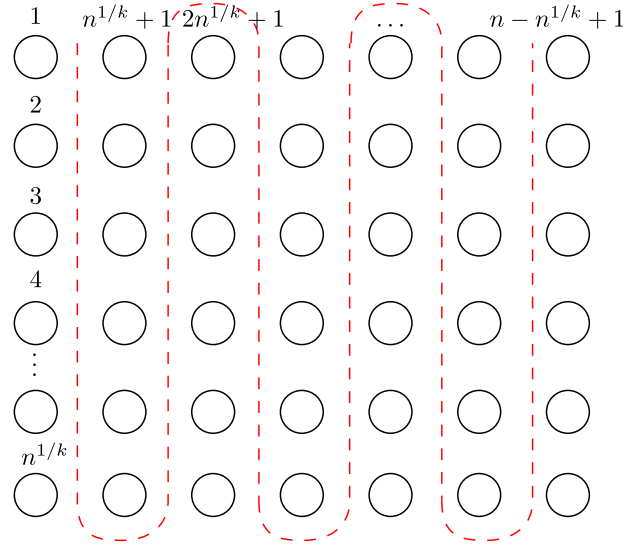


FIG. S2 (Color online). A snaking scheme to assign indices to qubits in 2D for a $n^{1/k} \times n^{1-1/k}$ grid, which is used in mapping to 1D.

This ensures that nearest-neighbor gates along one axis of the 2D grid map to nearest-neighbor gates in 1D. Gates along the other axis, however, correspond to nonlocal gates in 1D. The distance between two qubits that have to participate in a gate is now $O(Ln^{1/k})$, where the equivalent grid in 2D would be of size $n^{1/k} \times n^{1-1/k}$. We again use state transfer to move close to a far-off qubit and then perform a nearest-neighbor entangling gate. This time is set by the state transfer protocol, and is now $O(\min[n^{1/k}L, n^{(\alpha-1/2)/k}L^{\alpha-1/2}])$. For large $k = \Theta(1)$, this gives us the bound $O(\min[L^{1+\delta}, L^{\alpha-1/2+\delta}])$ for any $\delta > 0$. Notice, however, that faster hardness in 1D comes at a high cost—the effective number of qubits on which we implement a hard circuit is only $\Theta(n^{1/k})$.

This example of 1D is very instructive—it exhibits one particular way in which the complexity phase transition can happen. As we take higher and higher values of k , the hardness time would decrease, coming at the cost of a decreased number of effective qubits. This smoothly morphs into the easiness regime when $\alpha \rightarrow \infty$ since in this regime both transitions happen at $t = \Theta(L)$.

If the definition of hardness is more stringent (in order to link it to fine-grained complexity measures such as explicit quantitative lower bound conjectures), then the above mentioned overhead is undesirable. In this case we would adopt the first strategy to implement SWAPs and directly implement a random IQP circuit on all the n qubits. This would increase the hardness time by a factor n .

B. Hard-core limit

In the hard-core limit $V \rightarrow \infty$, the strategy is modified. Let us consider a physical qubit to represent the presence ($|1\rangle$) or absence ($|0\rangle$) of a boson at a site. A nearest-neighbor hop translates to a term in the Hamiltonian that can be written in terms of the Pauli operators as $XX + YY$. Further, an on-site field $J_{ii}a_i^\dagger a_i$ translates to a term $\propto Z$. There are no other terms available, in particular single-qubit rotations about other axes X or Y . This is because the total boson number is conserved, which in the spin basis corresponds to the conservation of $\sum_i Z_i$. This operator indeed commutes with both the allowed Hamiltonian terms specified above.

Let us now discuss the computational power of this model. When the physical qubits are constrained to have nearest-neighbor interactions in 1D, this model is nonuniversal and classically simulable. This can be interpreted due to the fact that this model is equivalent to matchgates on a path (i.e. a 1D nearest-neighbor graph), which is nonuniversal for quantum computing without access to a SWAP gate. Alternatively, one can apply the Jordan-Wigner transformation to map the spin model onto free fermions. One may then apply the fact that fermion sampling is simulable on a classical computer [2].

When the connectivity of the qubit interactions is different, the model is computationally universal for BQP. In the matchgate picture, this result follows from Ref. [49], which shows that matchgates on any graph apart from the path or the cycle are universal for BQP in an encoded sense. In the fermion picture, the Jordan-Wigner transformation

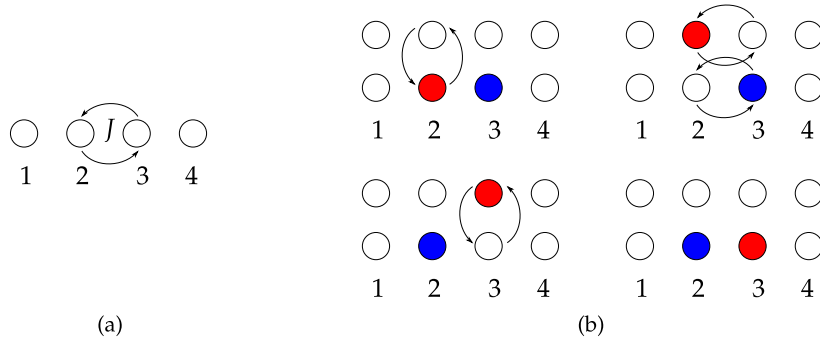


FIG. S3 (Color online). (a) A hopping between sites 2 and 3 that implements the mode unitary $\begin{pmatrix} \cos(|J|t) & -i \sin(|J|t)J/|J| \\ -i \sin(|J|t)J^*/|J| & \cos(|J|t) \end{pmatrix} = e^{-it(\text{Re}\{J\}X + \text{Im}\{J\}Y)}$. When $|J|t = \pi$, this is a SWAP between two modes with phases $(-iJ/|J|, -iJ^*/|J|)$ that depends on $\arg J$, the argument of J . (b) A “physical” SWAP between sites 2 and 3 by using ancilla sites available whenever the system is not nearest-neighbor in 1D. The colors are used to label the modes and how they move, and do not mean that both sites are occupied. The total hopping phase incurred when performing the physical SWAP can be set to be $(+i, -i)$, which cannot be achieved with just the hopping term shown in (a).

on any graph other than a path graph would typically result in nonlocal interacting terms that are not quadratic in general. Thus, the model cannot be mapped to free, quadratic fermions and the simulability proof from Ref. [2] breaks down.

Alternatively, a constructive way of seeing how we can recover universality is as follows. Consider again the dual rail encoding and two logical qubits placed next to each other as in Fig. S3. Apply a coupling $J(a_2^\dagger a_3 + a_3^\dagger a_2)$ on the modes 2 and 3 for time $t = \frac{\pi}{2J}$. This effects the transition $|10\rangle_{23} \rightarrow -i|01\rangle_{23}$ and $|01\rangle_{23} \rightarrow -i|10\rangle_{23}$, while leaving the state $|11\rangle_{23}$ the same. Now we swap the modes 2 and 3 using an ancilla mode that is available by virtue of having either long-range hopping or having $D > 1$. This returns the system back to the logical subspace of exactly one boson in modes 1 & 2, and one boson in modes 3 & 4, and effects the unitary $\text{diag}\{1, 1, 1, -1\}$ in the (logical) computational basis. This is an entangling gate that can be implemented in $O(1)$ time and thus the hardness timescale for hard-core interactions is the same as that of Hubbard interactions with $V = \Theta(1)$.

We finally discuss the case when V is polynomially large. Using the dual-rail encoding and implementing the same protocol as the non-hardcore case now takes the state $|11\rangle_{23} \rightarrow \lambda |11\rangle_{23} + \mu \frac{|20\rangle_{23} + |02\rangle_{23}}{\sqrt{2}}$, with $\mu \propto \frac{J}{\sqrt{8J^2 + V^2}} \sin\left(\frac{t\sqrt{8J^2 + V^2}}{2}\right)$. When $|\mu| \neq 0$, we get an error because the state is outside the logical subspace. The probability with which this action happens is suppressed by $1/V^2$, however, which is polynomially small when $V = \text{poly}(n)$.

However, one can do better: by carefully tuning the hopping strength $J \in [0, 1]$ and the evolution time t , one can always achieve the goal of getting $\mu = 0$ exactly and implementing an operation $\exp[-i\frac{\pi}{2}X]$ in the $|10\rangle_{23}, |01\rangle_{23}$ subspace. This requires setting $t\sqrt{2J^2 + \frac{V^2}{4}} = m\pi$ and $t = \frac{2\pi}{J}$ for integer m . This can be solved as follows: set $m = \lceil \sqrt{8 + V^2} \rceil$, and $J = \frac{V}{\sqrt{m^2 - 8}}$ (which is ≤ 1 since $m \geq \sqrt{8 + V^2}$). The time is set by the condition $t = \frac{2\pi}{J}$, which is $\Theta(1)$. This effects a logical CPHASE $[\phi]$ gate with angle $\phi = -\pi U/J$.

Finally, the above parameters that set μ exactly to zero work even for exponentially large $V = \Omega(\exp(n))$, but this requires exponentially precise control of the parameters J and t , which is not physically feasible. In this case, we simply observe that $|\mu|^2$, the probability of going outside the logical subspace and hence making an error, is $O(1/V^2)$, which is exponentially small in n . Therefore, in this limit, the gate we implement is exponentially close to perfect, and the complete circuit has a very small infidelity as well.

S4. HARDNESS TIMESCALE FOR FREE BOSONS

In this section, we review Aaronson and Arkhipov’s method of creating a linear optical state that is hard to sample from [10]. We then give a way to construct such states in time $\tilde{O}\left(m^{\alpha/D-1/2}\right)$ with high probability in the Hamiltonian model, and prove Theorem 2 for free bosons.

For free bosons, in order to get a state that is hard to sample from, we need to apply a Haar-random linear-optical unitary on m modes to the state $|1, 1, \dots, 1, 0, 0, \dots, 0\rangle$. Aaronson and Arkhipov gave a method of preparing the resulting state in $O(n \log m)$ depth in the circuit model. Their method involves the use of ancillas and can be thought of as implementing each column of the Haar-random unitary separately in $O(\log m)$ -depth. Here we mean that we apply the map $|1\rangle_j \rightarrow \sum_{i \in \Lambda} U_{ij} |1\rangle_i$ to “implement” the column i of the linear-optical unitary U . In the Hamiltonian model, we can apply simultaneous, non-commuting terms of a Hamiltonian involving a common site. The only constraint is that each term of the Hamiltonian should have a bounded norm of $1/d(i, j)^\alpha$. In this model, when α is small, it is possible to implement each unitary in a time much smaller than $O(\log m)$ —indeed, we show the following:

Lemma 5. *Let U be a Haar-random unitary on m modes. Then with probability $1 - \frac{1}{\text{poly}(m)}$ over the Haar measure, each of the first n columns of U can be implemented in time $O\left(\frac{\sqrt{\log m}}{m^{1/2-\alpha/D}}\right)$.*

To prove this, we will need an algorithm that implements columns of the unitary. For convenience, let us first consider the case $\alpha = 0$. The algorithm involves two subroutines, which we call the single-shot and state-transfer protocols. Both protocols depend on the following observation. If we implement a Hamiltonian that couples a site i to all other sites $j \neq i$ through coupling strengths J_{ij} , then the effective dynamics is that of two coupled modes a_i^\dagger and $b^\dagger = \frac{1}{\omega} \sum_{j \neq i} J_{ij} a_j^\dagger$, where $\omega = \sqrt{\sum_{j \neq i} J_{ij}^2}$. The effective speed of the dynamics is given by ω —for instance the time period of the system is $\frac{2\pi}{\omega}$.

The single-shot protocol implements a map $a_i^\dagger \rightarrow \gamma_i a_i^\dagger + \sum_{j \neq i} \gamma_j a_j^\dagger$. This is done by simply applying the Hamiltonian $H \propto a_i^\dagger (\sum_{j \neq i} \gamma_j a_j) + \text{h.c.}$ for time $t = \frac{1}{\omega} \cos^{-1} |\gamma_i|$. In the case $\alpha = 0$, we can set the proportionality factor equal to $1/\max |\gamma_j|$. This choice means that the coupling strength between i and the site k with maximum $|\gamma_k|$ is set to 1 (the maximum), and all other couplings are equal to $|\frac{\gamma_j}{\gamma_k}|$.

The other subroutine, the state-transfer protocol is also an application of the above observation and appears in Ref. [26]. It achieves the map $a_i^\dagger \rightarrow \gamma_i a_i^\dagger + \gamma_j a_j^\dagger$ via two rounds of the previous protocol. This is done by first mapping site i to the uniform superposition over all sites except i and j , and then coupling this uniform superposition mode to site j . The time taken for this is $\frac{1}{\omega} (\frac{\pi}{2} + \cos^{-1} |\gamma_i|)$. Since $\omega = \sqrt{m-2}$ (all $m-2$ modes are coupled with equal strength to modes i or j), this takes time $O\left(\frac{1}{\sqrt{m}}\right)$.

With these subroutines, we can describe the algorithm.

Algorithm 1: Algorithm for implementing one column of a unitary

Input: Unitary U , column index j

- 1 Reassign the mode labels for modes $i \neq j$ in nonincreasing order of $|U_{ij}|$.
 - 2 Implement the state-transfer protocol to map the state $a_j^\dagger |\text{vac}\rangle$ to $U_{jj} a_j^\dagger |\text{vac}\rangle + \sqrt{1 - |U_{jj}|^2} a_1^\dagger |\text{vac}\rangle$. Skip this step if $|U_{jj}| \geq |U_{j1}|$ already.
 - 3 Use the single-shot protocol between site 1 and the rest ($i \neq 1, j$) to map $a_1^\dagger \rightarrow \frac{U_{1j}}{\sqrt{1 - |U_{jj}|^2}} a_1^\dagger + \sum_{i \neq 1, j} \frac{U_{ij}}{\sqrt{1 - |U_{jj}|^2}} a_i^\dagger$.
-

It can be seen that Algorithm 1 implements a map $a_j^\dagger \rightarrow U_{jj} a_j^\dagger + \sum_{i \neq j} U_{ij} a_i^\dagger$, as desired. To prove Lemma 5 we need to examine the runtime of the algorithm when U is drawn from a Haar-random distribution.

Proof of Lemma 5. First, notice that since the Haar measure is invariant under the action of any unitary, we can in particular apply a permutation map to argue that the elements of the i 'th column are drawn from the same distribution as the first column. Next, recall that one may generate a Haar-random unitary by first generating m uniform random vectors in \mathbb{C}^m and then performing a Gram-Schmidt orthogonalization. In particular, this means that the first column of a Haar-random unitary may be generated by generating a uniform random vector with unit norm. This implies that the marginal distribution over any column of a unitary drawn from the Haar measure is simply the uniform distribution over unit vectors, since we argued above that all columns are drawn from the same distribution.

Now, let us examine the runtime. The first step (line 2 of the algorithm) requires time $t = O\left(\frac{1}{\sqrt{m}}\right)$ irrespective of U_{jj} because the total time for state-transfer is $\frac{1}{\omega} (\frac{\pi}{2} + \cos^{-1} U_{jj}) \leq \frac{\pi}{\omega} = \frac{\pi}{\sqrt{m-2}}$. Next, the second step takes time

$t = \frac{1}{\omega} \cos^{-1} \left(\frac{|U_{1j}|}{\sqrt{1-|U_{1j}|^2}} \right) = O(\frac{1}{\omega})$. Now,

$$\omega = \sqrt{1^2 + \frac{|U_{3j}|^2/(1-|U_{jj}|^2)}{|U_{2j}|^2/(1-|U_{jj}|^2)} + \frac{|U_{4j}|^2}{|U_{2j}|^2} + \dots} \quad (\text{S14})$$

$$= \sqrt{\frac{\sum_{i=2, i \neq j}^m |U_{ij}|^2}{|U_{2j}|^2}} = \sqrt{\frac{1-|U_{1j}|^2-|U_{jj}|^2}{|U_{2j}|^2}} \quad (\text{S15})$$

Now in cases where $|U_{jj}| \leq |U_{1j}|$ (where $|U_{1j}|$ is the maximum absolute value of the column entry among all other modes $i \neq j$), which happens with probability $1 - \frac{1}{m}$, we will have $\omega^2 \geq \frac{1-2|U_{1j}|^2}{|U_{2j}|^2}$. In the other case when $|U_{jj}| \geq |U_{1j}|$, meaning that the maximum absolute value among all entries of column j is in row j itself, we again have $\omega^2 \geq \frac{1-2|U_{jj}|^2}{|U_{2j}|^2}$. Both these cases can be written together as $\omega^2 \geq \frac{1-2|U_{1j}|^2}{|U_{2j}|^2}$, where we now denote U_{1j} as the entry with maximum absolute value among *all* elements of column j . The analysis completely hinges on the typical ω we have, which in turn depends on $|U_{1j}|$. We will show $\Pr \left(\omega^2 \geq \frac{cm}{\log m} \right) \geq 1 - \frac{1}{\text{poly}(m)}$, which will prove the claim for $\alpha = 0$.

$$\Pr \left(\omega^2 \geq \frac{cm}{\log m} \right) \geq \Pr \left(1 - 2|U_{1j}|^2 \geq c_1 \ \& \ |U_{2j}|^2 \leq \frac{c_1 \log m}{cm} \right) \quad (\text{S16})$$

since the two events on the right hand side suffice for the first event to hold. Further,

$$\Pr \left(1 - 2|U_{1i}|^2 \geq c_1 \ \& \ |U_{2j}|^2 \leq \frac{c_1 \log m}{cm} \right) \geq \Pr \left(|U_{1j}|^2 \leq \frac{c_1 \log m}{cm} \right) \quad (\text{S17})$$

for large enough m with some fixed $c_1 = 0.99$ (say), since $|U_{2j}|^2 \leq |U_{1j}|^2$ and $1 - 1.98 \log m/m \geq 0.99$ for large enough m .

To this end, we refer to the literature on order statistics of uniform random unit vectors $(z_1, z_2, \dots, z_m) \in \mathbb{C}^m$ [50]. This work gives an explicit formula for $F(x, m)$, the probability that all $|z_j|^2 \leq x$. We are interested in this quantity at $x = c_1 \log m / (cm)$, since this gives us the probability of the desired event ($\omega^2 \geq cm / \log m$). We have

$$\Pr \left(\frac{1}{k+1} \leq x \leq \frac{1}{k} \right) = \sum_{l=0}^k \binom{m}{l} (-1)^l (1-lx)^{m-1}. \quad (\text{S18})$$

It is also argued in Ref. [50] that the terms of the series successively underestimate or overestimate the desired probability. Therefore we can expand the series and terminate it at the first two terms, giving us an inequality:

$$\Pr \left(\frac{1}{k+1} \leq x \leq \frac{1}{k} \right) = 1 - m(1-x)^{m-1} + \frac{m^2}{2}(1-2x)^{m-1} - \dots \quad (\text{S19})$$

$$\geq 1 - m(1-x)^{m-1}. \quad (\text{S20})$$

Choosing $c = c_1/4 = 0.2475$, we are interested in the quantity when $k = \lfloor \frac{m}{4 \log m} \rfloor$:

$$\Pr(x \leq 4 \log m/m) \geq 1 - m(1 - 4 \log m/m)^{m-1} \geq 1 - \frac{1}{m^{3-4/m}}, \text{ since} \quad (\text{S21})$$

$$(1 - 4 \log m/m)^{m-1} = \exp \left[(m-1) \log \left(1 - \frac{4 \log m}{m} \right) \right] \leq \exp \left[-4(m-1) \frac{\log m}{m} \right] = m^{-4(1-1/m)}. \quad (\text{S22})$$

This implies that the time for the single-shot protocol is also $t = O(\frac{1}{\omega}) = O(\sqrt{\frac{\log m}{m}})$ for a single column. Notice that we can make the polynomial appearing in $\Pr(\omega^2 \geq cm / \log m) \geq 1 - 1/\text{poly}(m)$ as small as possible by suitably reducing c . To extend the proof to all columns, we use the union bound. In the following, let t_j denote the time to

implement column j .

$$\Pr\left(\exists j : t_j > \sqrt{\frac{\log m}{cm}}\right) \leq \sum_j \Pr\left(t_j > \sqrt{\frac{\log m}{cm}}\right) \quad (\text{S23})$$

$$\leq m \times \frac{1}{\text{poly}(m)} = \frac{1}{\text{poly}(m)} \quad (\text{S24})$$

when the degree in the polynomial is larger than 1, just as we have chosen by setting $c = 0.2475$. This implies

$$\Pr\left(\forall j : t_j \leq \sqrt{\frac{\log m}{cm}}\right) = 1 - \Pr\left(\exists j : t_j > \sqrt{\frac{\log m}{cm}}\right) \geq 1 - \frac{1}{\text{poly}(m)}. \quad (\text{S25})$$

This completes the proof in the case $\alpha = 0$. When $\alpha \neq 0$, we can in the worst-case set each coupling constant to a maximum of $O(m^{-\alpha/D})$, which is the maximum coupling strength of the furthest two sites separated by a distance $O(m^{1/D})$. This factor appears in the total time for both the state-transfer [26] and single-shot protocols, and simply multiplies the required time, making it $O\left(\sqrt{\frac{\log m}{m}} \times m^{\alpha/D}\right) = O\left(\frac{\sqrt{\log m}}{m^{1/2-\alpha/D}}\right)$. Finally, if there are any phase shifts that need to be applied, they can be achieved through an on-site term $J_{ii}a_i^\dagger a_i$, whose strength is unbounded by assumption and can thus take arbitrarily short time. \square

The total time for implementing boson sampling on n bosons is therefore $O\left(n \frac{\sqrt{\log m}}{m^{1/2-\alpha/D}}\right) = O\left(n^{1+\beta(\frac{\alpha}{D}-\frac{1}{2})}\right)$, since we should implement n columns in total.

A. Optimizing hardness time

We can optimize the hardness time by implementing boson sampling not on n bosons, but on n^δ of them, for any $\delta \in (0, 1]$. The explicit lower bounds on running time of classical algorithms we would get assuming fine-grained complexity-theoretic conjectures is again something like $\exp\left[n^{\text{poly}(\delta)}\right]$ for any $\delta \in (0, 1]$. This grows very slowly with n , but it still qualifies as subexponential, which is not polynomial or quasipolynomial (and by our definition would fall in the category “hard”).

The idea is to do boson sampling on the nearest set of n^δ bosons. The linear extent of the sublattice in which these reside is given by $Ln^{\delta/D}$, and the number of sites in this sublattice is $m_{\text{eff}} = L^D n^\delta$. Since L grows polynomially with n when $\beta > 1$, the number of effective sites is much larger than the number of effective bosons, as desired for boson sampling. If $\beta = 1$, we choose an effective number of modes $m_{\text{eff}} = \Theta(n^{2\delta})$, as is required for hardness.

Using the protocol in Lemma 5, the total time to implement n^δ columns of an $m_{\text{eff}} \times m_{\text{eff}}$ linear optical unitary is $n^\delta \min[m_{\text{eff}}^{\alpha/D-1/2}, m_{\text{eff}}^{1/D}] = n^{O(\delta)} \min[L^{\alpha-D/2}, L]$, both for $\beta = 1$ and $\beta > 1$. This gives the hardness exponent in Theorem 2 for free bosons. When we compare with Ref. [16], which states a hardness result for $\alpha \rightarrow \infty$, we see that we have almost removed a factor of n from the timescale coming from implementing n columns of the linear optical unitary. This gives us a hardness timescale $L^{1+\delta}$ that almost matches the easiness timescale of L . More importantly, this makes the noninteracting hardness timescale the same as the interacting one.